Summerschool Around the Zilber-Pink conjectures

25 June - 5 July 2012, Paris

Abstracts

Introductory courses

Antoine CHAMBERT-LOIR

Heights on abelian varieties

The height machine is the tool that allows to perform arguments in the style of Fermat's infinite descent in the context of algebraic varieties over number fields. During the XXth century, it has been proved an invaluable tool for the study of number theoretical questions such as the Mordell-Weil theorem (structure of the group of points of Abelian varieties over number fields or the (now proven) Mordell conjecture and its generalizations. It also lead to interesting questions in Diophantine Geometry, for example the Lehmer problem or the Bogomolov conjecture.

The lectures will review these problems and results, emphasizing the parts of the arguments involving heights, providing en passant the necessary background concerning the algebraic geometry of abelian varieties.
Generalizations to other algebraic groups (semi-abelian varieties) or to dynamical systems (algebraic dynamics) will be evoked if time permits.

Christophe CORNUT and Bruno KLINGLER
Shimura varieties

This course will give an introduction to Shimura varieties. It will cover the following topics: Hodge theory, background on algebraic groups, Mumford-Tate groups, symmetric spaces and arithmetic groups, Shimura data, Shimura varieties as moduli spaces, special subvarieties, Hecke correspondences, canonical models.

Thomas SCANLON

Tutorial on o-minimality

In this tutorial, conducted in coordination with Sergei Starchenko and Margaret Thomas, we shall reprise the basics of the theory of o-minimality and then expose some of the applications of the Pila-Wilkie counting theorem to questions around the Zilber-Pink conjectures. A rough distribution of the topics to be covered in the various lectures follows.

Lecture 1 (TS): The basic theory of o-minimality with a discussion of the fundamental examples of divisible ordered abelian groups and real closed fields and a detailed sketch of the cell decomposition theorem.

Lecture 2 (TS): An account of quantifier elimination and o-minimality for $\mathbb{R}_{an}$ (the real field together with restricted analytic functions), $\mathbb{R}_{\exp}$ (the real field with the real exponential function), and $\mathbb{R}_{an,\exp}$ (the amalgam of these two structures).

Lecture 3 (SS): O-minimal complex analysis (see Starchenko's abstract)

Lecture 4 (MT): The Pila-Wilkie parametrization theorem (see Thomas' abstract)

Lecture 5 (TS): The Pila-Wilkie counting theorem: Using the parametrization theorem proven in Lecture 4 and some diophantine estimates we establish the theorem of Pila-Wilkie bounding the number of rational points in definable sets.

Lecture 6 (TS): Pila-Zannier method: We explain some of the applications of the counting theorem to the André-Oort conjecture and related Zilber-Pink problems.

Plenary talks

Yuri BILU

Effectivity and uniformity in the Theorem of André (after Lars Kühne)

I'll speak on effective and uniform versions of the André theorem, due to Lars Kühne. A typical result: subject all special points on a non-special plain curve can be effectively bounded, and (subject to some restrictions) their number can be bounded only in terms of the degree of the curve and of the degree of the definition field.

Emmanuel BREUILLARD

Heights on character varieties of reductive groups and the uniform Tits alternative

A natural notion of normalized height can be defined on the character variety of a reductive group, which extends the classical notion of height on tori. The Lehmer problem can be addressed for this height. It turns out that when the group is semisimple, the analogue of the Lehmer conjecture holds for this height. In fact the height of a point is bounded away from zero by a positive constant. This height gap theorem has applications to geometric group theory, in particular in establishing uniform bounds on the growth rate of linear groups and in proving a uniform version of the Tits alternative.

Maria CARRIZOSA

From Lehmer to Zilber-Pink

In this talk we explain how the relative Lehmer problem is related to the Zilber-Pink conjecture. The idea comes from the first article of Bombieri, Masser and Zannier on the subject: they used the lower bounds for the height given by the Lehmer problem to prove that the bounded height subset of $\mathbb{G}_m^n$ that they were interested in, was finite. Indeed, Lehmer type bounds are used to prove that such a subset has bounded degree and by Northcott property they concluded finiteness. We'll explain how this argument works in the context of abelian varieties.

Philipp HABEGGER

Torsion Points on the Weierstrass Family of Elliptic Curves

Pairs of complex numbers (a, b) outside the vanishing locus of $4a^3 + 27b^2$ parametrize elliptic curves in Weierstrass form $y^2 = x^3 + ax + b$. If we fix the first coordinate to be 1, then the set of (a, b) such that $(1, \sqrt{1 + a + b})$ is torsion on the corresponding elliptic curve can be quite big. Indeed, the curve given by $1 + a + b = 0$ gives rise to infinitely many points of order 2. However, Masser and Zannier asked if there are only finitely (a, b) where the three points $(1, *)$, $(2, *)$, and $(3, *)$ are simultaneously torsion. They had proved a finiteness result for two points on the (one parameter) Legendre family of elliptic curves.

The answer to Masser and Zannier's question is yes. I will discuss a proof in two steps. The first half uses the Pila-Wilkie Theorem as in the strategy proposed by Zannier to treat diophantine problems of Manin-Mumford type. The second half consists of a bound for the torsion on an elliptic curve by David and a height inequality."

Patrik HUBSCHMID

André-Oort conjecture in positive characteristic

In this talk, we consider an analogue of the André-Oort conjecture in positive characteristic. It states that an irreducible subvariety of a Drinfeld modular variety containing a Zariski dense subset of special points is a special subvariety.

We will first introduce Drinfeld modular varieties and explain the notion of special subvariety in positive characteristic. Then we will explain how the methods of Edixhoven, Klingler and Yafaev in the classical case can be adapted to our case. This leads to a proof of the conjecture for special points with separable reflex field over the base field. Finally, I will provide an outlook about possible methods to tackle the case of inseparable reflex fields using equidistribution statements for Hecke orbits.

Bruno KLINGLER

André-Oort conjecture modulo GRH I and II

I will explain the proof of the André-Oort conjecture under GRH, using the ergodic and Galois-theoretic results explained in Ullmo's and Yafaev's lectures.

Guillaume MAURIN

Application de la méthode de Vojta au cas torique de la conjecture de Zilber-Pink

La méthode de Vojta repose sur une inégalité de hauteurs qui implique assez directement des propriétés de hauteur bornée pour les intersections de sous-groupes et de sous-variétés. Les travaux de Vojta, Faltings, Bombieri,. . . , Rémond montrent comment déduire une telle inégalité d'une minoration de nombres d'intersection présentant une certaine uniformité relativement à divers paramètres. Dans cet exposé, nous expliquerons comment démontrer cette minoration dans le cas torique.

Nicolas RATAZZI

Lehmer's problem on abelian varieties and multiplicative groups

Points of height zero are torsion points; for the multiplicative group this is a classical result of Kronecker, for abelian varieties this is due to Néron and Tate. Lehmer's problem and its analogs search for optimal lower bounds for the height of non torsion points. We will discuss the techniques and related results useful for Zilber-Pink type conjectures on abelian varieties and multiplicative groups.

Gaël RÉMOND

Application de la méthode Vojta au cas abélien de la conjecture de Zilber-Pink.

J'expliquerai comment la méthode introduite par Vojta pour donner une deuxième d'émonstration de la conjecture de Mordell et généralisée ensuite par Faltings pour résoudre le probléme de Mordell-Lang peut être également appliquée au cas abélien de la conjecture de Zilber-Pink. Le point de départ consiste à voir celle-ci comme une famille de problèmes de Mordell-Lang simultanés dans divers quotients de la variété abélienne initiale.

Jonathan PILA

Some special point problems

I will discuss some "special point" problems involving elliptic modular surfaces. Examples of the latter include the Legendre surface $L : y^2 = x(x-1)(x-\lambda)$ considered as a family of elliptic curves fibered over the λ-line. A special point of L is a torsion point on a CM fibre. The "mixed" André-Oort conjecture for L was proved by André (2001); I will outline a proof for L^n. The proof follows the method used to prove AO for products of modular curves using o-minimality, point-counting, and a suitable "Ax-Lindemann" statement, itself proved using o-minimality. I will present some further generalisations. The o-minimality assures that the results come with a certain uniformity which I will use to give an analogue for singular moduli of Mann's theorem on vanishing sums of roots of unity.

Damian RÖSSLER

On the ramification of torsion points lying on curves of genus at least two

We give effective bounds for the degree of ramification of torsion points lying on curves of genus > 1 over local fields . The speaker was led to the computation of these bounds while he was trying to apply Hrushovski's technique of proof of the Manin-Mumford conjecture to a ramified situation, where the involved Galois polynomials are not cyclotomic anymore.

Sergei STARCHENKO

Complex analysis in o-minimal structures

In this talk we consider holomorphic functions and complex analytic manifolds definable in an o-minimal structure. We will show that o- minimality implies strong theorems on removal of singularities and survey some of these results. In addition we will also discuss the definability in o-minimal structures of several classical holomorphic maps, and some corollaries concerning definable families of abelian varieties.

This is joint work with Kobi Peterzil.

Margaret THOMAS

The Pila-Wilkie theorem

We will examine the proof of the parameterization theorem of Pila and Wilkie, a geometric result for bounded sets and functions definable in o-minimal expansions of real closed fields.

Emmanuel ULLMO

André-Oort conjecture : ergodic tools

This talk will present ergodic tools and equidistribution results on Shimura varieties with a view towards the proof of André-Oort conjecture under GRH.

Emmanuel ULLMO

La conjecture d'Ax-Lindeman hyperbolique pour les variétés de Shimura projectives et applications à la conjecture d'André-Oort

On formulera la conjecture d'Ax-Lindemann hyperbolique et sa place dans la stratégie de Pila-Zannier pour une preuve inconditionnelle de la conjecture d'André-Oort. Le résultat principal est la conjecture d'Ax-Lindemann hyperbolique pour les variétés de Shimura projectives (travail en commun avec A. Yafaev). Nous expliquerons comment en déduire une preuve de la conjecture d'André-Oort pour les sous-variétés de Shimura projectives d'une puissance arbitraire du module des variétés abéliennes principalement polarisées de dimension 6.

Andrei YAFAEV

André-Oort conjecture : Hecke orbits

This talk will discuss Hecke operators action and Galois orbits on Shimura varieties with a view towards the proof of André-Oort conjecture under GRH.

Umberto ZANNIER

Unlikely intersections with compact subgroups (in dimension 2)

The topic of the talk arises from the Manin-Mumford conjecture and its extensions, where we shall focus on the case of (complex connected) commutative algebraic groups G of dimension 2. The 'Manin-Mumford' context in these cases predicts finiteness for the set of torsion points in an algebraic curve inside G, unless the curve is of 'special' type, i.e. a translate of an algebraic subgroup of G.

We shall consider not merely the set of torsion points, but its topological closure in G (which turns out to be also the maximal compact subgroup). In the case of abelian varieties this closure is the whole space, but this is not so for other G; actually, we shall prove that in certain cases (where a natural dimensional condition is fulfilled) the intersection of this larger set with a non-special curve must still be a finite set.

We shall conclude by stating in brief some extensions of this problem to higher dimensions.

Boris ZILBER

Generalised special subvarieties and atypical intersections

I will discuss a very general definition of a special subvariety of X^n, for an algebraic variety X, inspired by recent works on those for Shimura varieties by Pila, Ullmo, Yafaev and others. The main theorem provides a classification of special subvarieties, reducing the possible geometric type to one of the three cases in the classification of Zariski geometries.
This also allows to formulate a very general Schanuel-type conjecture and, consequently, a Diophantine conjecture on atypical intersections.

Satellite meeting

Martin BAYS

An introduction to model-theoretic issues around Zilber-Pink

Zilber was led to what was to become the Zilber-Pink conjecture as part of his study of the model theory of structures arising from complex analysis. I will sketch this context, describing from a distance the main examples and concepts.

Juan Diego CAYCEDO

Applications of the Weak CIT

Zilber's Conjecture on Intersections with Tori (CIT), the toric case of the Zilber-Pink conjecture, implies that the statement of Schanuel's conjecture can be expressed by a set of first-order sentences in the language of exponential fields. It also implies the first-order expressibility of analogous Schanuel-style inequalities for some related structures: fields with raising to powers relations and green fields. In these latter cases, however, the use of the conjecture can be replaced by applications of a weak version of the conjecture, known as the Weak CIT, which is known to follow from Ax's theorem (differential field version of Schanuel's conjecture), and Laurent's theorem (Mordell-Lang for algebraic tori). I will present the argument giving unconditional first-order expressibility. In the case of raising to powers this is due to Zilber.

Ayhan GÜNAYDIN

Rational solutions of polynomial-exponential equations

We consider polynomial-exponential equations over complex numbers where the variables run through rational numbers. We present a method to reduce the rational solutions to integer ones and give a description of them using the earlier results. As a corollary, we get a finiteness result.

Adam HARRIS

Categoricity of the j invariant and arithmetic geometry

We will look at how the model theoretic notion of categoricity of a natural structure involving the j invariant translates into arithmetic geometry.

Jonathan KIRBY

Zilber-Pink and axiomatising exponentiation

One of the key conjectures about complex exponentiation is Schanuel's conjecture, and it is also an axiom for Zilber's exponential field. I will explain how the relevant case of Zilber-Pink is equivalent to (an appropriate statement of) Schanuel's conjecture being expressible in first-order logic. The key is viewing Zilber-Pink as a uniformity statement about algebraic subgroups. This was Zilber's original motivation for coming up with the conjecture. (Joint work with Boris Zilber)

Vincenzo MANTOVA

A pseudoexponentiation-like structure on the algebraic numbers

When constructing pseudoexponentiation, Zilber introduced a notion of "exponential-algebraic closure" for exponential fields, stating that certain systems of exponential-polynomial equations must have solutions. It is possible to construct an exponential-algebraic closed field whose elements are just the algebraic numbers, and such that the kernel of the exponential function is cyclic. In some ways, it still resembles complex and pseudo exponentiation, but on the other hand any statement of Schanuel type is false. The problem of finding an exponential field of this kind is mostly an arithmetic one, and its solution is obtained by controlling the likely intersections inside the product of copies of the additive and of the multiplicative group.

Tamara SERVI

On the interdefinability of Weierstrass $\wp$-functions

Bianconi (1997) proved that the real exponential function and the sine function restricted to a bounded interval are not interdefinable. The proof has two main ingredients: the model-completeness and o-minimality of the structures generated by the two mentioned functions, and a theorem of Ax (1971) on a Schanuel condition for power series over the complex field.
We prove the following: let $f_0, f_1, \ldots, f_n$ be Weierstrass $\wp$-functions; then f_0 is locally definable from $f_1, \ldots, f_n$ and the exponential function if and only if f_0 is obtained from one of $f_1, \ldots, f_n$ by isogeny or Schwarz reflection.

The proof uses a result of Wilkie (2007) on local definability of holomorphic functions and results of Kirby (2009), in the spirit of Ax's theorem (joint work with Gareth Jones and Jonathan Kirby).

Margaret THOMAS

Counting rational and algebraic points on definable sets

Pila and Wilkie's theorem, concerning the density of rational and algebraic points lying on sets definable in o-minimal expansions of the real field, has already had far-reaching consequences for diophantine geometry. Wilkie has conjectured an improvement to their main result for sets definable in the real exponential field. We shall survey some results that have been obtained in this direction, including the proven one-dimensional case of the conjecture, some partial results for certain surfaces, and some applications.