

# Lecture 1. Introduction to André-Oort and Zilber-Pink.

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Let us start with some set-theoretic generalities on the André-Oort type questions. Suppose we are given an algebraic variety  $X$  over  $\overline{\mathbb{Q}}$  which is endowed with a set  $S(X)$  of irreducible subvarieties (called special subvarieties, special subvarieties of dimension zero are called special points) defined over  $\overline{\mathbb{Q}}$  such that

1.  $X \in S(X)$
2. For any  $Z \in S(X)$ , the set of special points  $x \in Z$  is Zariski dense in  $Z$ .
3. Components of intersections of special subvarieties are special.

In particular, the third condition implies that for any subset  $W$  of  $X(\mathbb{C})$ , there exists a smallest special subvariety  $Z_W$  containing  $W$ .

Examples of this situation are :

1. Abelian varieties and subvarieties in  $S(X)$  are translates of abelian subvarieties by torsion points.
2. Tori and products of subtori by points of finite order.
3. Shimura varieties and special subvarieties.

.. also the mixed situations : mixed Shimura varieties and semi-abelian varieties...

**Conjecture 0.1 (Abstract André-Oort conjecture)** *The following equivalent statements hold :*

1. *An irreducible component of the Zariski closure of a set of special points is special.*

2. For any algebraic subvariety  $Y$  of  $X$ , there exists a finite set  $Z_1, \dots, Z_r$  of special subvarieties of  $Y$  such that for any special subvariety  $Z \subset Y$ ,

$$Z \subset \cup_{i=1}^r Z_i$$

In other words there is a **maximal** set of special subvarieties in  $Y$ .

Let us prove the equivalence. Suppose (1) holds. Let  $\Sigma$  be the set of all special points contained in  $Y$  and write

$$\Sigma^{Zar} = \cup_{i=1}^r Z_i$$

By assumption (1),  $Z_i$  are special subvarieties of  $Y$ . Clearly any special subvariety of  $Y$  is contained in the union of the  $Z_i$  (because it contains a dense set of special points).

Suppose (2) holds. Let  $\Sigma$  be a set of special points and let  $Y$  be a component of its Zariski closure. Replace  $\Sigma$  with  $\Sigma \cap Y$ . Let  $Z_i$ s be as in (2). Then  $\Sigma \subset \cup Z_i$  but as  $\Sigma$  is Zariski dense in  $Y$ , we see that  $Y$  is one of the  $Z_i$ .

**Définition 0.1** A sequence  $(Z_n)$  of special subvarieties is called **strict** if for any proper **special** subvariety  $Z$ ,

$$\{n \in \mathbb{N} : Z_n \subset Z\} \text{ is finite}$$

A sequence  $(Z_n)$  of special subvarieties is called **generic** if for any proper subvariety  $Y$ ,

$$\{n \in \mathbb{N} : Z_n \subset Y\} \text{ is finite}$$

Of course any generic sequence is strict. We can now reformulate the conjecture 0.1 as follows.

**Proposition 0.2** The conjecture 0.1 is equivalent to the following statement :

**Any strict sequence of special points is generic.**

**Proof.** Suppose that the conclusion of 0.1 holds. Let  $(x_n)$  be a strict sequence and suppose there is a subvariety  $Y$  such that  $\{n : x_n \in Y\}$  is infinite. The Zariski closure of  $\{x_n : x_n \in Y\}$  is contained in  $Y$  and has a positive dimensional component which is special. This contradicts the assumption that  $(x_n)$  is strict. Hence  $(x_n)$  is generic.

Suppose that any strict sequence is generic. Let  $\Sigma$  be a set of special points and let  $Y$  be a component of  $\Sigma^{Zar}$ . Replace  $\Sigma$  by  $\Sigma \cap Y$ . Let  $Z_Y$  be the smallest special subvariety containing  $Y$ . We claim that we can find a strict sequence in  $Z_Y$ . Let  $S(Z_Y)$  be the set of proper special subvarieties of  $Z_Y$ . As they are defined over  $\overline{\mathbb{Q}}$ , this is a countable set. Let us number them :

$$S(Z_Y) = \{Z_n : n \in \mathbb{N}\}$$

Consider

$$\Sigma_n := \{x \in \Sigma : x \notin \cup_{i=1}^n Z_i\}$$

As  $\Sigma$  is Zariski dense in  $Y$ , we can choose  $x_n \in \Sigma_n$ . The sequence  $(x_n)$  is strict in  $Z_Y$ . By assumption,  $(x_n)$  is generic and therefore, as  $Y$  contains infinitely many elements of the sequence,  $Y = Z_Y$  and is special.  $\square$

## 1 Preliminaries on equidistribution.

The situation becomes much more interesting when  $X(\mathbb{C})$  and subvarieties in  $S(X)$  are endowed with a canonical probability measure. This will indeed be the case in the cases we are going to deal with.

Let  $X$  be a metric space and  $P(X)$  the set of Borel probability measures on  $X$ . Let  $C(X)$  be the set of *bounded* continuous functions on  $X$ .

**Définition 1.1** *We say that the sequence  $\mu_n \in P(X)$  is weakly convergent to  $\mu \in P(X)$  if for all  $f \in C(X)$*

$$\mu_n(f) = \int_X f d\mu_n \longrightarrow \mu(f) = \int_X f d\mu \text{ as } n \longrightarrow \infty$$

*One writes*

$$\mu_n \longrightarrow \mu$$

*One defines the weak\* topology on  $P(X)$  to be the smallest topology that makes the maps  $\mu \longrightarrow \mu(f) = \int f d\mu$  (for  $f \in C(X)$ ) continuous.*

**Définition 1.2 (Equidistribution)** *Let  $X$  and  $S(X)$  be as before. We suppose that any  $Z \in S(X)$  is canonically endowed with a probability measure  $\mu_Z$  such that the closure of  $\text{Supp}(\mu_Z)$  is  $Z$ .*

*We say that a sequence  $(Z_n)$  with  $Z_n \in S(X)$  is **equidistributed** if there exists a  $Z \in S(X)$  such that after possibly extracting a subsequence the following conditions hold :*

1.  $\mu_{Z_n} \longrightarrow \mu_Z$  (weakly)
2.  $Z_n \subset Z$  for all  $n$  large enough.

We have the following :

**Proposition 1.1** *Suppose that  $Z_n$  and all its subsequences are equidistributed. Then components of the Zariski closure of  $\cup_n Z_n$  are special.*

**Proof.** Easy and left as exercise. □

In fact, a stronger statement than 0.1 is the following, so-called **equidistribution conjecture**.

Let  $X/\overline{\mathbb{Q}}$  and  $S(X)$  as before. Let  $F$  be a number field of definition of  $X$ . To any  $x \in X(\overline{\mathbb{Q}})$ , one associates the following measure :

$$\Delta_x = \frac{1}{[F(x) : F]} \sum_{y \in \text{Gal}(\overline{F}/F)x} \delta_y$$

where  $\delta$  denotes the Dirac measure.

**Conjecture 1.2 (Equidistribution conjecture)** *Let  $x_n$  be a strict sequence of special points in  $X$ . Then*

$$\Delta_n \longrightarrow \mu_X \text{ ( weakly )}$$

**Proposition 1.3** *The conjecture 1.2 implies 0.1.*

**Proof.** Let  $\Sigma$  be a set of special points,  $Y$  a component of its Zariski closure. Replace  $\Sigma$  by  $\Sigma \cap Y$  and let  $Z_Y$  be the smallest special subvariety containing  $Y$ . We can also assume that  $Y$  is defined over  $F$  (enlarge  $F$  if necessary).

Replace  $X$  by  $Z_Y$ . As before, one extracts a strict subsequence  $(x_n)$  from  $\Sigma$ . Then  $\Delta_{x_n}$  converges to  $\mu_Z$ . But, as  $Y$  is defined over  $F$ ,

$$\text{Supp}(\Delta_n) \subset Y$$

By passing to the limit, we find that  $\text{Supp}(\mu_Z) \subset Y$ . As  $Y$  is closed, we get  $Z \subset Y$  hence  $Y = Z$  and  $Y$  is special. □

The Equidistribution conjecture is known for tori (Bilu), abelian varieties (Clozel-Ullmo-Zhang). In the case of Shimura varieties, it is only known for modular curves in full generality (Duke). Zhang proves equidistribution for some Hilbert modular varieties but only for sequences satisfying an additional assumption.

Equidistribution conjecture is one of the directions for future developments. One has to prove two things : one is the equidistribution of *toric* orbits and then derive from it the equidistribution of Galois orbits. There are some recent results on equidistribution of toric orbits by Eiseindler, Linderstrauss, Venkatesh, Ph. Michel...

## 2 Strategy of the proof.

The strategy of the proof of the André-Oort conjecture is as follows.

Let  $(Z_n)$  be a sequence of special subvarieties and let  $Y$  be a component of the Zariski closure of  $\cup_n Z_n$ . Replace  $(Z_n)$  by the subsequence contained in  $Y$ . We need to prove that  $Y$  is special.

Choose a number field  $F$  such that  $Y$  is defined over  $F$ . In the cases we consider,  $X$  is canonically endowed with an ample line bundle  $\mathcal{L}$  and we can calculate degrees of subvarieties with respect to this line bundle.

Then **one of the following occurs** :

1. The sequence  $(Z_n)$  is equidistributed and so are all its subsequences. In this case  $Y$  is special and we are done.
- 2.

$$\deg_{\mathcal{L}}(\text{Gal}(\overline{F}/F) \cdot Z_n) \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

In this case we will be able to construct for each  $n$  large enough a special subvariety  $Z'_n$  such that

$$Z_n \subset Z'_n \subset Y$$

and  $\dim(Z'_n) > \dim(Z_n)$ .

Reiterate the process with  $Z'_n$  instead of  $Z_n$ . In the end we will get that  $Y$  is special.

### 3 Further generalisation : Zilber-Pink.

**Conjecture 3.1 (Zilber-Pink)** *Let  $Z \subset S$  be an irreducible closed subvariety. The intersection of  $Z$  with the union of all special subvarieties of dimension  $< \dim(S_Z) - \dim(Z)$  is not Zariski dense in  $Z$ .*

This is of course equivalent to

**Conjecture 3.2** *Suppose that  $Z$  is Hodge generic (i.e.  $S_Z = S$ ). The intersection of  $S$  with the union of all special subvarieties of codimension  $> \dim(Z)$  is not Zariski dense in  $Z$ .*

Zilber-Pink conjecture certainly implies André-Oort. Indeed, suppose that  $Z$  is not special i.e.  $\dim(S_Z) - \dim(Z) > 0$ . Then by Zilber-Pink, the set of special points lying on  $Z$  is not Zariski dense.

# Lecture 2. Lower bounds for Galois degrees and the alternative.

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Our aim in this lecture is to give lower bounds for degrees of Galois orbits of special subvarieties and characterise sequences of special subvarieties whose Galois degree is uniformly bounded. This lies at the heart of the proof.

Whenever possible and appropriate we refer to Ullmo's lecture notes that contain complete proofs.

Our aim is to obtain a lower bounds for degrees of Galois orbits of special subvarieties in terms of certain invariants attached to the Shimura data defining special subvarieties in question.

## 1 Lower bounds for the Galois degree of special subvarieties.

We start with some preliminaries.

As usual we consider a Shimura datum  $(G, X)$  and a compact open subgroup  $K$  of  $G(\mathbb{A}_f)$ .

First of all, we assume  $G$  to be semisimple of adjoint type and we work with the connected component

$$S = \Gamma \backslash X^+$$

the image of  $X^+ \times \{1\}$  in  $\mathrm{Sh}_K(G, X)$  where  $X^+$  is a fixed connected component of  $X$ . We suppose that  $K$  is neat and that  $K = \prod_p K_p$  with  $K_p$  compact open subgroups of  $G(\mathbb{Q}_p)$ . These assumptions do not change the shape of the lower bounds for the Galois orbits we are looking for.

Indeed :

The natural morphism  $\pi: G \longrightarrow G^{\text{ad}}$  induces a morphism of Shimura data  $(G, X) \longrightarrow (G^{\text{ad}}, X^{\text{ad}})$ . Choosing a compact open subgroup  $K^{\text{ad}}$  of  $G^{\text{ad}}(\mathbb{A}_f)$  containing  $\pi(K)$ , we obtain a morphism of Shimura varieties :

$$\text{Sh}_K(G, X) \longrightarrow \text{Sh}_{K^{\text{ad}}}(G^{\text{ad}}, X^{\text{ad}})$$

This morphism has finite fibres and is defined over the reflex field  $E(G, X)$ , hence it is enough to study Galois orbits in  $\text{Sh}_{K^{\text{ad}}}(G^{\text{ad}}, X^{\text{ad}})$ . We replace  $(G, X, K)$  by  $(G^{\text{ad}}, X^{\text{ad}}, K^{\text{ad}})$ .

Next we replace, if necessary,  $K$  a neat compact open subgroup of finite index which is a product. Again, this changes nothing to the shape of the Galois orbits we obtain.

Lastly, to move to the component  $S$ , we apply a suitable Hecke correspondence (see lemma 2.35 of Ullmo's notes) and this does not affect the shape of the orbits either.

One shows that a special subvariety  $V$  of  $S$  is a component of the image of  $\text{Sh}_{K \cap H(\mathbb{A}_f)}(H, X_H)$  where  $(H, X_H)$  is a Shimura subdatum with  $H$  the generic Mumford-Tate group on  $X_H$ . (note that no Hecke correspondence is involved!) This is lemma 2.36 of Ullmo's notes. We usually identify  $V$  with this component (i.e. we forget about the map from  $\text{Sh}_{K_H}(H, X_H)$  to  $\text{Sh}_K(G, X)$ ).

One proves a certain number of uniformity results. First of all, as  $H = TH^{\text{der}}$  ranges through the set of special subvarieties of  $S$ ,  $T \cap H^{\text{der}}$  is uniformly bounded. We assume that  $T$  is a nontrivial torus. By definition,  $V$  is a non-strongly special subvariety.

Let  $C = H/H^{\text{der}}$ . This is a non-trivial torus, isogeneous to  $T$  by an isogeny of uniform degree  $m$  ( $m = |T \cap H^{\text{der}}|$ ). Let  $L = L_C = L_T$  be the splitting field of  $C$ . From any  $x \in X$ , we get a special Shimura datum  $(C, \{x\})$  and a surjective reciprocity morphism  $r_C: L^* \longrightarrow C$ . The construction of such a morphism is explained in Ullmo's lectures, section 2.2.9. The morphism  $r_C^m$  factors through  $T \longrightarrow C$  and hence gives a surjective morphism

$$r: L^* \longrightarrow T$$

where  $L^* = \text{Res}_{L/\mathbb{Q}} \mathbb{G}_{mL}$ . As  $T$  varies, the degree of  $L$  is bounded hence there is a finite number of possibilities for  $\text{Gal}(L/\mathbb{Q})$ . We can therefore assume that as an abstract group this group is fixed. By numbering the elements we get a basis for  $X^*(L^*) = \mathbb{Z}[\text{Gal}(L/\mathbb{Q})]$ . We can identify  $X^*(T)$  with a submodule of  $X^*(L^*)$  via  $r^*$  and hence calculate the coordinates of elements in  $X^*(T)$  with



respect to our chosen basis. Fix a faithful representation  $\rho: G \rightarrow \mathrm{GL}(V)$ . This induces a representation of  $T$ . There exists a finite subset  $\mathcal{X}$  of  $X^*(T)$  such that the representation corresponds to the grading

$$V_L = \bigoplus_{\chi \in \mathcal{X}} V_\chi$$

where  $V_\chi$  is the subspace on which  $T$  acts via  $\chi$ . One proves

**Proposition 1.1** *The coordinates (with respect to the basis as above) of the characters  $\chi \in \mathcal{X}$  are uniformly bounded.*

The weights in the representation of  $T$  induced by  $\rho$  are uniformly bounded.

The proofs of these uniformity results are technical and we omit them here.

We now define the degree of Galois orbits :

**Definition 1.1 (Degree of Galois orbits of special subvarieties.)** *Let  $V$  be a special subvariety of  $S$ . Let  $\bar{S}$  be its Baily-Borel compactification. We define  $\deg(\mathrm{Gal}(\bar{F}/F) \cdot V$  to be the degree of the Zariski closure in  $\bar{S}$  of the subvariety  $\mathrm{Gal}(\bar{F}/F) \cdot V$  of  $S$  calculated with respect to the Baily-Borel bundle on  $S$ .*

Let us state the main theorem :

**Theorem 1.2** *We keep the above notations.*

*Let  $F$  be the field of definition of the canonical model of  $S$ . There exists a constant  $C > 0$ ,  $B > 0$  and  $\mu > 0$  such that the following holds.*

*Let  $V \subset S$  be a special subvariety defined by a Shimura datum  $(H, X_H)$  and  $T$  be the connected centre of  $H$ . We let  $K_T^m$  be the compact open subgroup of  $T(\mathbb{A}_f)$  and  $K_T := T(\mathbb{A}_f) \cap K$ . We let  $i(T)$  be the number of primes such that  $K_{T,p}^m \neq K_{T,p}$ . We let  $d_L$  be the absolute value of the discriminant of the splitting field of  $L$ .*

*Then*

$$\deg_{\mathcal{L}_K} \mathrm{Gal}(\bar{F}/F) \cdot V > CB^{i(T)} |K_T^m / K_T| d_L^\mu$$

Note that in the case where  $V$  is a special point, this degree is simply the number of geometric components of  $\mathrm{Gal}(\bar{F}/F) \cdot V$  i.e. the number of Galois conjugates of  $V$ .

First of all, recall that the morphism

$$\mathrm{Sh}_{K_H}(H, X_H) \longrightarrow \mathrm{Sh}_K(G, X)$$

extends to compactifications and for any subvariety  $Z$  of  $\mathrm{Sh}_{K_H}(H, X_H)$ . We denote by  $\mathcal{L}_H$  the Baily-Borel line bundle on  $\mathrm{Sh}_{K_H}(H, X_H)$  and by  $\mathcal{L}_K$ , the Baily-Borel line bundle on  $\mathrm{Sh}_K(G, X)$ .

One shows that

$$\mathrm{deg}_{\mathcal{L}_H}(Z) \leq \mathrm{deg}_{\mathcal{L}_K}(Z)$$

(we identify  $Z$  with its image). This is a fairly difficult proposition and we omit the proof here.

In view of this inequality, it suffices to give a lower bound for

$$\mathrm{deg}_{\mathcal{L}_H}(Z)$$

Let  $V$  be a special subvariety. As before,  $V$  is a component of the image of  $\mathrm{Sh}_{K_H}(H, X_H)$  (where  $K_H := H(\mathbb{A}_f) \cap K$ ).

Let  $K_T^m$  be the unique maximal compact open subgroup of  $T(\mathbb{A}_f)$ . Let us define  $K_H^m := K_T^m K_H$ , this is a compact open subgroup of  $H(\mathbb{A}_f)$  containing  $K_H$ . One needs to alter the group  $K_H^m$  a little bit : replace it with  $K_{H,3}^m \cap K_3 \prod_{p \neq 3} K_{H,p}^m$  to make it neat (we suppose that  $K_3$  is the congruence subgroup of level three). This only changes the constants by a uniform integer.

The morphism

$$\mathrm{Sh}_{K_H}(H, X_H) \longrightarrow \mathrm{Sh}_{K_H^m}(H, X_H)$$

is finite of degree  $|K_H^m/K_H| = |K_T^m/K_T|$ .

The group  $K_H^m/K_H = K_T^m/K_T$  acts on  $\mathrm{Sh}_{K_H^m}(H, X_H)$ .

**We break the degree of the Galois orbit into two pieces and estimate each piece separately. The estimation of the first piece will not involve GRH, while the second one relies on the GRH.**

**Lemma 1.3** *The degree of  $\mathrm{Gal}(\overline{F}/F) \cdot V$  is at least the degree of  $\mathrm{Gal}(\overline{F}/F) \cdot V \cap \pi^{-1}\pi(V)$  times the number of Galois conjugates of  $\pi(V)$ .*

**Proof.** The proof is based on the fact that the degree of components of  $\pi^{-1}\sigma\pi(V)$  (for  $\sigma \in \mathrm{Gal}(\overline{F}/F)$ ) does not change.

Let us set  $V^m = \pi(V)$ . Let  $V_i$  be a component of  $\pi^{-1}(\sigma V^m)$ ,  $\sigma V$  is one of them and  $\alpha V_i = \sigma V$  for some  $\alpha \in K_T^m/K_T$ . However, by a fundamental property of the Baily-Borel line bundle,  $\alpha^* \mathcal{L}_H = \mathcal{L}_H$  and hence

$$\deg_{\mathcal{L}_H}(V_i) = \deg_{\mathcal{L}_H}(\sigma V) = \deg_{\mathcal{L}_H}(V)$$

We see that :

$$\deg_{\mathcal{L}_H}(\text{Gal}(\overline{F}/F) \cdot V \cap \pi^{-1} \sigma \pi(V)) = \deg_{\mathcal{L}_H}(\sigma V) |\text{Irr}(\text{Gal}(\overline{F}/F) \cdot V \cap \pi^{-1} \sigma V^m)|$$

and

$$|\text{Irr}(\text{Gal}(\overline{F}/F) \cdot V \cap \pi^{-1} \sigma V^m)| = |\text{Irr}(\text{Gal}(\overline{F}/F) \cdot V \cap \pi^{-1} V^m)|$$

□

Let us first deal with the second piece. This is where GRH is used.

Recall some facts about the Galois action.

Following Deligne we define for any reductive  $\mathbb{Q}$ -group  $N$

$$\pi(N) := N(\mathbb{A})/N(\mathbb{Q})\rho(\tilde{N}(\mathbb{A})).$$

Here  $\rho: \tilde{N} \rightarrow N^{\text{der}}$  denotes the universal covering of  $N^{\text{der}}$ . Let

$$\overline{\pi}_0(\pi(N)) := \pi_0(\pi(N))/\pi_0(N(\mathbb{R})_+),$$

then

$$\pi_0(H, K_H) = \overline{\pi}_0(\pi(H))/K_H = H(\mathbb{A}_f)/H(\mathbb{Q})_+ K_H.$$

Let  $E_H$  be the reflex field of  $E_H$ .

The action of  $\text{Gal}(\overline{\mathbb{Q}}/E_H)$  on  $\pi_0(H, K_H)$  is given by the reciprocity morphism

$$r_{(H, X_H)}: \text{Gal}(\overline{\mathbb{Q}}/E_H) \rightarrow \overline{\pi}_0(\pi(H))$$

The morphism  $r_{(H, X_H)}$  factors through  $\text{Gal}(\overline{\mathbb{Q}}/E_H)^{\text{ab}}$  which is identified via global class field theory with  $\pi_0(\pi(E_H^*))$  where  $E_H^* = \text{Res}_{E_H/\mathbb{Q}} \mathbb{G}_{mE_H}$ .

To  $(H, X_H)$  one associates two Shimura data  $(C, \{x\})$  and  $(H^{\text{ad}}, X_{H^{\text{ad}}})$ . The field  $E_H$  is the composite of  $E(C, \{x\})$  and  $E(H^{\text{ad}}, X_{H^{\text{ad}}})$ . There are morphisms of Shimura data

$$\theta^{\text{ab}}: (H, X_H) \rightarrow (C, \{x\}) \text{ and } \theta^{\text{ad}}: (H, X_H) \rightarrow (H^{\text{ad}}, X_{H^{\text{ad}}}).$$

Note that  $(C, \{x\})$  is a special Shimura datum. Let  $r_C := r_{(C, \{x\})}$  be the reciprocity morphism associated with  $(C, \{x\})$ . The morphism  $\theta^{\text{ab}}$  induces a morphism  $\overline{\pi}_0(\pi(H)) \rightarrow \overline{\pi}_0(\pi(C))$ . This morphism preceded by  $r_{(H, X_H)}$  is  $r_{(C, \{x\})}$ . We let  $F$  be the Galois closure of  $E_H$ . Note that the degree of  $F$  over  $\mathbb{Q}$  is bounded uniformly on  $(H, X_H)$ .

The number of conjugates of  $\pi(V)$  is the size of the image of  $\text{Gal}(\overline{F}/F)$  in  $\overline{\pi}_0(\pi(H))/K_H^m$  by  $r_H$  which is at least the size of the image of  $r_C((\mathbb{A}_f \otimes F)^*)$  (where  $C = H/H^{\text{der}}$ ) in  $\overline{\pi}_0(\pi(C))/K_C^m$ . Let  $J$  be this image, we show :

**Proposition 1.4** *Assume GRH for CM fields.*

*We have*

$$|J| \gg d_L^\mu$$

where  $\epsilon$  depends on  $(G, X)$  only.

Recall that  $L$  is the splitting field of  $C$  or  $T$ ,  $d_L$  the absolute value of the discriminant of  $L$ .

Note that in the original paper, Ullmo-Yafaev, the bound is a power of logarithm which is enough for the purposes of proving the André-Oort conjecture. Here we provide a better bound, obtained in a recent paper : Ullmo-Yafaev ‘Nombres de classes de tores de multiplication complexe’.

**Proof.** Recall that we have a surjective morphism of tori  $r_C: L^* \rightarrow C$ .

It suffices to give a lower bound for the image of  $r_C((\mathbb{A}_f \otimes L)^*)$  in  $C(\mathbb{Q}) \backslash C(\mathbb{A}_f)/K_C^m$ .

**The idea is that we will generate a sufficiently large subgroup of  $C(\mathbb{Q}) \backslash C(\mathbb{A}_f)/K_C^m$  by taking the images of adèles lying over small split prime whose existence is guaranteed by the GRH.**

For an abelian group  $H$  and an integer  $l$ , we define  $M_H(l)$  to be the smallest integer  $A$  such that for any  $l$ -tuple  $(g_1, \dots, g_l)$  of elements of  $H$ , there exists  $(a_1, \dots, a_l) \in \mathbb{Z}^n \setminus \{0\}$  with  $\sum_j |a_j| \leq A$  satisfying  $g_1^{a_1} \cdots g_l^{a_l} = 1$ .

Let  $l = |H|$  and let  $g_1, \dots, g_l \in H$ . If  $g_i = 1$  for some  $i$ , then we have a non-trivial multiplicative relation involving the  $g_i$ s with  $A = 1$ . Otherwise we have a relation  $g_i g_j^{-1} = 1$  for  $i \neq j$ .

In all cases, we have a relation  $g_1^{a_1} \cdots g_l^{a_l} = 1$  with  $\sum |a_i| \leq 2$ . We see that for  $l = |H|$ , we have

$$M_H(l) \leq 2.$$

Let  $h_L = L^* \backslash (L \otimes \mathbb{A}_F)^* / (\widehat{\mathbb{Z}} \otimes O_L)^*$  be the class group of  $L$  and let us take  $H = h_L / \ker(r_C)$ .

We will show the following inequality

$$M_H(l) > c \frac{\log(d_L)}{\log(l) + \log \log(d_L)} \quad (1)$$

where  $c > 0$  is a uniform constant.

This implies the estimation of  $|H|$  :

$$|H| > \frac{d_L^{c/2}}{\log(d_L)} \gg d_L^\mu$$

with uniform  $\mu > 0$ .

We will now show the inequality (1). Recall some uniformity results :

The reciprocity morphism  $r_C: L^* \rightarrow C$  induces an inclusion  $X^*(C) \subset X^*(L^*)$  and we have a canonical basis of  $X^*(L^*)$  given by numbering the elements of  $\text{Gal}(L/\mathbb{Q})$ . There is a basis  $\mathcal{B}$  of  $X^*(C)$  such that the coordinates of  $\chi \in \mathcal{B}$  with respect to our chosen basis of  $X^*(L^*)$  are uniformly bounded. As  $G$  is adjoint, for all characters  $\chi$  of  $\mathcal{B}$ ,  $\chi\bar{\chi}$  is the trivial character.

Let  $l \geq 1$  be an integer and  $p_1, \dots, p_l$ ,  $l$  primes splitting  $C$  and  $a_1, \dots, a_l$  be elements of  $\mathbb{Z}$ . For each  $i$ , fix a place  $v_i$  of  $L$  above  $p_i$  and an idele  $P_i$  in  $(L \otimes \mathbb{A}_f)^*$  which is the uniformiser at the place  $v_i$  and 1 elsewhere. Consider  $I = P_1^{a_1} \cdots P_r^{a_r} \subset (L \otimes \mathbb{A}_f)^*$  and its image  $\bar{I}$  in  $h_L$ .

Suppose that  $\bar{I}$  is in the kernel of  $\bar{r}$  i.e

$$r_C(I) = \pi k$$

where  $\pi \in C(\mathbb{Q})$  et  $k \in K_C^m$ . Let  $\pi_i = \chi_i(\pi) \subset L^*$ .

One shows that  $\mathbb{Q}[\pi_1, \dots, \pi_r] = L$ . This uses that the primes  $p_i$  are split in a crucial way!

Let  $t$  be a uniform bound on coordinates of the characters  $\chi_i$ . We see that  $\pi'_i := (p_1^{|a_1|} \cdots p_l^{|a_l|})^t \pi_i \in O_L$  and the fact that  $\chi_i \bar{\chi}_i$  is the trivial character implies that

$$|\sigma(\pi'_i)| \leq (p_1^{|a_1|} \cdots p_l^{|a_l|})^{2t}$$

for all  $\sigma \in \text{Gal}(L/\mathbb{Q})$ .

Let  $n_L$  be the degree of  $L$  over  $\mathbb{Q}$ . As  $L$  is a decomposition field of a torus, of fixed dimension  $d$ ,  $n_L$  is uniformly bounded.

We choose a basis  $b_1, \dots, b_{n_L}$  of  $L$  over  $\mathbb{Q}$  (as a vector space) with  $b_k = \prod_{i=1}^d \pi_i^{n_{i,k}}$  for integers  $n_{i,k}$  such that  $n_{i,k} \leq n_L$ .

The facts that the  $b_i$  are in  $O_L$  and that  $b_1, \dots, b_{n_L}$  form a basis of  $L$  over  $\mathbb{Q}$ , imply that  $\mathbb{Z}[b_1, \dots, b_{n_L}]$  is an order in  $O_L$ . In particular

$$|\text{Discr}(\mathbb{Z}[b_1, \dots, b_{n_L}])| \geq D_L \quad (2)$$

On the other hand  $|\text{Discr}(\mathbb{Z}[b_1, \dots, b_{n_L}])|$  is the determinant of the matrix  $(\text{Tr}_{L/\mathbb{Q}}(b_i b_j))$ . By Hadamard inequality, if  $b$  is an upper bound for the  $|\text{Tr}_{L/\mathbb{Q}}(b_i b_j)|$ , then

$$|\text{Discr}(\mathbb{Z}[b_1, \dots, b_{n_L}])| \leq c(n_L) b^{n_L}$$

where  $c(n_L)$  depends on  $n_L$  only (one can take  $c(n_L) = n_L^{n_L}$ ).

The fact that  $|\sigma(\pi'_i)| \leq (p_1^{a_1} \dots p_l^{a_l})^{2A}$ , shows that there is a uniform integer  $D$  (depending on  $n_L$  only) such that

$$|\text{Tr}_{L/\mathbb{Q}}(b_i b_j)| \leq (p_1^{a_1} \dots p_l^{a_l})^D$$

hence after replacing  $D$  by  $Dn_L$ , we get

$$|\text{Discr} \mathbb{Z}[b_1, \dots, b_{n_L}]| \leq c(n_L) (p_1^{a_1} \dots p_l^{a_l})^{Dn_L}$$

Let  $c = 1/c(n_L)$ . The equation (2) gives :

$$(p_1^{a_1} \dots p_l^{a_l})^D \geq cD_L$$

We will now choose  $l$  and the  $p_i$ .

Let us recall the effective Cheboarev theorem.

**Theorem 1.5** *Assume GRH. Let  $\pi_L(x)$  be the number of primes  $p$  split in  $L$  such that  $p \leq x$ . There exist absolute constants  $c_1 > 0$  and  $c_2 > 0$  such that*

$$\pi_L(x) \geq c_2 \frac{x}{\log(x)}$$

for all  $x \geq c_1 \log(D_L)^2 (\log \log D_L)^4$ .

Let  $l \geq 1$  be a prime and

$$x = c_3 l \log(l) + c_1 \log(D_L)^2 (\log \log D_L)^4$$

where  $c_3$  is a uniform constant that we will make explicit below.

We want  $c_3$  to be such that  $c_2 \frac{x}{\log(x)} \geq l$ . Note that (if  $x \geq e$ )

$$\frac{x}{\log(x)} \geq \frac{c_3 l \log(l)}{\log(c_3) + 2 \log(l)}.$$

Then we have  $c_2 \frac{x}{\log(x)} \geq l$  as soon as  $\frac{c_2 c_3}{\log(c_3) + 2} \geq 1$ . We can take  $c_3 = \max(e, \frac{4}{c_2^2})$ .

Using effective Chebotarev theorem, we take  $p_1, \dots, p_l$  split in  $L$  and satisfying

$$p_i \leq x.$$

Let  $I$  be as before Let  $A = \sum_{i=1}^l |a_i|$ . We have

$$AD \log(x) \geq \log(cD_L)$$

On the other hand, there is a uniform  $c_5$  such that

$$\log(x) \leq c_5(\log(l) + \log \log(D_L))$$

We obtain a lower bound for  $A$  of required form. This finishes the proof of (1) and of the theorem. □

**We now go back to the first piece.**

Consider again the morphism  $\pi: \text{Sh}_{K_H}(H, X_H) \longrightarrow \text{Sh}_{K_H^m}(H, X_H)$  and recall that the group  $K_T^m/K_T$  acts on  $\text{Sh}_{K_H}(H, X_H)$ .

Let  $\Theta$  be the image of  $r((\widehat{\mathbb{Z}} \otimes O_L)^*)$  in  $K_T^m/K_T$ . Then  $\Theta$  contains the image of the morphism  $x \mapsto x^a$  (where  $a$  is some uniform integer) on  $K_T^m/K_T$ . Replace  $\Theta$  by this group (image of  $x \mapsto x^a$ ). We have

$$|\text{Irr}(\pi^{-1}\pi(V))| = |\text{Irr}((K_T^m/K_T)V)| \leq |K_T^m/K_T|/|\Theta| |\text{Irr}(\Theta V)|$$

The kernel of the map  $x \mapsto x^a$  is bounded by  $D^{i(T)}$  where  $i(T)$  is the number of primes  $p$  such that  $K_{T,p}^m \neq K_{T,p}$  (use that  $K_T^m/K_T$  is a finite abelian group with a set of generators of uniformly bounded cardinality - embed  $T$  into a finite and uniform number of copies of  $L^*$ ). We have

$$|K_T^m/K_T|/|\Theta| \geq D^{i(T)}$$

Let us consider  $|\text{Irr}((K_T^m/K_T)V)|$ .

Let  $Z := \pi^{-1}V^m$ , we get, because  $\pi^*\mathcal{L}_{K_H^m} \cong \mathcal{L}_{K_H}$ ,

$$\deg_{\mathcal{L}_{K_H}}(Z) = \deg_{\mathcal{G}_{\pi^*\mathcal{L}_{K_H^m}}}(Z) = \deg_{\mathcal{L}_{K_H^m}}(\pi_*Z) = [K_T : K_T^m] \deg_{\mathcal{L}_{K_H^m}}(V^m) \geq [K_T : K_T^m].$$

**Notice here that we bounded below  $\deg_{\mathcal{L}_{K_H^m}}(V^m)$  by 1**

On the other hand, one has

$$\deg_{\mathcal{L}_{K_H}}(Z) \leq \deg_{\mathcal{L}_K}(Z) = \deg(Z).$$

We deduce that

$$\deg(Z) \geq [K_T : K_T^m].$$

We have see that

$$\deg_{\mathcal{L}_{K_H}}(\text{Gal}(\bar{F}/F) \cdot V \cap \pi^{-1}V^m) = \deg_{\mathcal{L}_{K_H}}(V) \cdot |\text{Irr}(\text{Gal}(\bar{F}/F) \cdot V \cap \pi^{-1}V^m)|$$

And, setting  $B = 1/D^{i(T)}$ , we get

$$\deg_{\mathcal{L}_{K_H}}(\text{Gal}(\bar{F}/F) \cdot V \cap \pi^{-1}\pi(V)) \geq B^{i(T)}|K_T^m/K_T|.$$

## 2 Sequences of special subvarieties with bounded Galois degree.

Our next task is to analyse the conditions under which a sequence  $Z_n$  associated to  $(H_n, X_n)$  is such that the degrees of the Galois orbits of  $Z_n$  are bounded.

Fix a non-trivial subtorus  $\subset G$  with  $T(\mathbb{R})$  compact. A sequence of special subvarieties  $Z_n$  defined by Shimura data  $(H_n, X_n)$  is called  $T$ -special if for each  $n$ ,  $H_n = TH_n^{der}$ .

In this section we show that sequences  $Z_n$  with bounded Galois degree split into a finite number of  $T$ -special subsequences.

Suppose this is the case. Using our lower bound, we first see that  $d_L$  is bounded, hence  $L$  can be assumed to be constant. By writing the decomposition

$$V = \bigoplus_{\chi} V_{\chi}$$

and using the fact that  $\chi$  lie in the fixed subset of  $X^*(T) \subset \mathbb{Z}[\text{Gal}(L/\mathbb{Q})]$  (identification via  $X^*(r)$ ), we see that the tori  $T$  lie in a finitely many  $\text{GL}_n(L)$



orbits and a Galois cohomology argument shows that in fact they lie in finitely many  $\mathrm{GL}_n(\mathbb{Q})$ -orbits.

**Suppose that the tori  $T$  lie in one  $\mathrm{GL}_n(\mathbb{Q})$ -conjugacy class.**

Let us now consider the term in  $B^{i(T)}|K_T^m/K_T|$ .

This term is bounded hence both  $i(T)$  and  $|K_T^m/K_T|$  are bounded. The group  $K_T^m/K_T$  is a finite product of groups  $K_{T,p}^m/K_{T,p}$ . For a prime  $p$  such that  $K_{T,p}^m = K_{T,p}$  one can show that the tori in question lie in one  $\mathrm{GL}_n(\mathbb{Z}_p)$  conjugacy class.

For a  $p$  such that  $K_{T,p}^m \neq K_{T,p}$  and  $p$  **unramified in  $L$** , lemma below shows that the tori lie in a finite number of  $\mathrm{GL}_n(\mathbb{Z}_p)$ -conjugacy classes.

**Lemma 2.1 (Clozel's lemma)** *Let  $G$  be a reductive group over  $\mathbb{Q}_p$ ,  $T \subset G$  a non trivial torus and let  $H = Z_G(T)$ . Let  $K$  be a fixed compact open subgroup of  $G(\mathbb{Q}_p)$  and let  $K_T = K_T^m$  be the maximal compact subgroup of  $T(\mathbb{Q}_p)$ . The function*

$$I(g) = |K_T/T(\mathbb{Q}_p) \cap g^{-1}Kg| \rightarrow \infty$$

*as  $g \rightarrow \infty$  in  $G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$  (where a basis of neighbourhoods of  $\infty$  is given by the complements of compact subsets of  $G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$ ). Let  $W$  be a set of  $g \in G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$  such that  $I(g)$  is bounded. The image of  $W$  in  $G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$  is finite.*

**We can assume that the tori  $T$  lie in one  $\mathrm{GL}_n(\mathbb{Z}) = \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{GL}_n(\widehat{\mathbb{Z}})$ -conjugacy class.**

We now need to pass from  $\mathrm{GL}_n(\mathbb{Z})$  to  $\Gamma$ . Because  $\Gamma$  has finite index in  $G(\mathbb{Z})$ , it suffices to prove that the tori lie in finitely many  $G(\mathbb{Z})$  conjugacy classes.

Let  $S$  be the finite set of primes  $p$  such that either  $T_{\mathbb{Z}_p}$  is not a torus or the closure of  $G$  in  $\mathrm{GL}_{n\mathbb{Z}_p}$  is not reductive and smooth. Let  $A := \mathbb{Z}_S$ . As the work in one  $\mathrm{GL}_n(\mathbb{Z})$ -conjugacy class, the tori we work with have smooth Zariski closure in  $\mathrm{GL}_{n\mathbb{Z}_S}$ . By a result of Gille-Moret Bailly, there are only finitely many  $G(A)$  conjugacy classes of such tori and Clozel's lemma finishes the proof.

**What we have seen is that after possibly extracting a subsequence, we can assume that the tori  $T$  lie in one  $\Gamma$ -conjugacy class, i.e. the sequence consists of  $T$ -special subvarieties.**

We have proved :

**Theorem 2.2** *Let  $Z_n$  be a sequence such that the Galois degree of  $Z_n$  is bounded. There exists a finite set of subtori  $\{T_1, \dots, T_r\}$  of  $G$  such that  $Z_n$  is  $T_i$ -special for some  $i$ .*

*Hence by a consequence of the theorem of Clozel-Ullmo,  $Z_n$  is equidistributed.*

# Lectures 3 and 4. Geometric criterion and proof of the André-Oort conjecture under GRH.

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In these last two lectures we give a sketch of the proof of the André-Oort conjecture assuming the GRH. In order to complete the proof, we need two major ingredients that we will explain in these lecture:

1. The geometric criterion involving Hecke correspondences.
2. The existence of suitable Hecke correspondences.

Let us recall the general situation (from the first lecture). We have our Shimura variety  $S$  and  $Z$  is a subvariety containing a Zariski dense sequence  $Z_n$  of special subvarieties. We denote by  $(H_n, X_n)$  the Shimura data defining  $Z_n$ . We let  $T_n$  be the connected centre of  $H_n$ . We assume that the  $Z_n$  are not  $T$ -special,  $T_n$ s are non-constant, non-trivial tori. We let  $L_n$  be the splitting field of  $T_n$  and  $d_n$  the absolute value of its discriminant. Our aim is to show that for all  $n \gg 0$ ,  $Z_n$  is contained in a special subvariety  $Z'_n \subset Z$  with  $\dim(Z'_n) > \dim(Z_n)$ .

## 1 The geometric criterion.

Section 2.4 of Ullmo's notes contains a statement showing that if  $Z$  is a *Hodge generic* subvariety of  $S$  and  $q \in G(\mathbb{Q})$  which is in  $K_p$  for all  $p$  except  $p = l$  for a certain prime  $l$  *large enough (depending on  $Z$ )* and such that  $T_q$ -orbits are dense and  $Z \subset T_q Z$ , then  $Z$  is special.

This criterion is used to prove the André-Oort conjecture in the case of curves, however, it is unusable in the higher dimensional cases. The dependence of  $l$  on  $Z$  is very non-effective and we will see in the end of this section

why this statement fails without the assumption ‘ $l$  large enough’. One may notice that in the case where  $S = \mathbb{C}^n$  (see Ullmo’s lecture), one can establish the dependence of  $l$  on the degree of  $Z$ . Here we present a refined version of this criterion.

Let  $(G, X)$  be a Shimura datum and  $(H, X_H)$  a **non-strongly special** subdatum. Write  $H = TH^{der}$  with  $T$  a non-trivial subtorus.

**Theorem 1.1 (Geometric criterion)** *Let  $(G, X)$  be a Shimura datum with  $G$  semisimple of adjoint type and let  $K$  be a neat compact open subgroup of  $G(\mathbb{A}_f)$  which is a product. We let, as usual,  $S$  be the ‘neutral’ component of  $\text{Sh}_K(G, X)$ .*

*Let  $V$  be a non-strongly special subvariety of  $S$  contained in a **Hodge-generic** subvariety  $Y$  of  $S$ . Let  $T$  be the connected centre of the group defining  $T$ .*

*Let  $l$  be a prime number splitting  $T$  and  $m$  an element of  $T(\mathbb{Q}_l)$ .*

*Suppose that  $Y$  satisfies the conditions*

- (1)  $Y \subset T_m Y$ .
- (2) for every  $k_1$  and  $k_2$  in  $K_l$  the image of  $k_1 m k_2$  in  $G(\mathbb{Q}_l)$  generates an unbounded (for the  $l$ -adic topology) subgroup of  $G(\mathbb{Q}_l)$ .

*Then  $Y$  contains a special subvariety  $V'$  containing  $V$  properly.*

**Proof.** We only present a sketch of the proof and omit all technicalities.

We refer to Ullmo’s notes, section 2.3 for the notion of monodromy.

Fix a smooth Hodge generic point  $x \in Y$  and fix  $V_{\mathbb{Z}}$  a faithful representation of  $G$ . Let

$$\rho: \pi_1(Y^{sm}, x) \longrightarrow \text{GL}(V_{\mathbb{Z}})$$

be the monodromy representation and let  $\Gamma'$  be its image. Recall (see Ullmo’s notes, Lemme 2.42) that  $\Gamma'$  is Zariski dense in (the image of)  $G$ .

We let  $K'_l$  be the closure of  $\Gamma'$  in  $G(\mathbb{Q}_l)$ . By Ullmo’s notes, Corollaire 2.44, the group  $K'_l$  is a compact open subgroup of  $K_l$ .

Let  $\text{Sh}^l(G, X)$  be the pro- $l$ - Shimura covering. This is the projective limit over compact open subgroups of  $G(\mathbb{A}_f^l)$ . Notice that  $G(\mathbb{Q}_l)$  acts on  $\text{Sh}^l(G, X)$ .

There is a component  $\tilde{Y}$  of the preimage of  $Y$  in  $\text{Sh}^l(G, X)$  which is stable by  $K'_l$ .

The condition  $Y \subset T_m Y$  implies that  $\tilde{Y}$  is also stable by  $k_1 m k_2$  for some  $k_1, k_2 \in K_l$ .

Hence  $Y$  is stable by the group  $U_l$  generated by  $K'_l$  and  $k_1mk_2$ . By assumption, this group is unbounded.

One then shows that there is a component  $V'$  of closure of the orbit of  $V$  by  $U_l$  which is a special subvariety containing  $V$  properly.  $\square$

The problem is to guarantee that for any  $k_1$  and  $k_2$ ,  $k_1mk_2$  generates an unbounded subgroup of  $G(\mathbb{Q}_l)$ . The idea is to replace the group  $K_l$  by a smaller one but whose index we can still control. We will assume that  $K_l$  by a suitable *Iwahori* subgroup.

Let  $I_l$  be an **Iwahori subgroup** in *good position with respect to  $T$* .

We are not going into details about what a Iwahori subgroup is, just list the relevant properties:

1.  $I_l \cap T(\mathbb{Q}_l)$  is the maximal compact subgroup of  $T(\mathbb{Q}_l)$ .

We will say that  $I_l$  is in 'good position with respect to  $T$ '.

2. The relation  $T_{m^r} = (T_m)^r$  holds in the Hecke algebra attached to  $I_l$ . This property will be used in an essential way.

The important property is that if  $T_{\mathbb{Q}_l}$  is split,  $T(\mathbb{Q}_l) \cap K_l = K_T^m$ , then there exists an Iwahori subgroup  $I_l$  of  $K_l$  in good position with respect to  $T$  and

$$|K_l/I_l| < l^f$$

where  $f$  is a uniform integer - independent of  $T$  !

Without the assumption that  $K_l$  is contained in a suitable Iwahori subgroup, it is actually quite easy to find examples of  $m$  such that  $T_m$  has dense orbits but there exist  $k_1$  and  $k_2$  such that  $k_1mk_2$  does not generate an unbounded subgroup.

Consider the following counter-example:  $G = \mathrm{PGL}_2$  and suppose  $K'_l$  is the group  $\Gamma_0(l)$ . The element  $m$  is  $m = \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix}$

Suppose that  $k_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $k_2 = 1$ .

One then sees that  $k_1mk_2$  has order two and will not generate an unbounded subgroup.

By assuming that  $K_l$  lies in a suitable Iwahori, which in this case is  $\Gamma_0(l)$ , we prevent this from happening: the element  $k_1$  is not in  $\Gamma_0(l)$ .

In his paper ‘Special points on products of two modular curves’, Edixhoven constructs counter-examples to the property ‘ $Z \subset T_q Z$  implies  $Z$  special’ in the case where  $S = \mathbb{C} \times \mathbb{C}$ . In his counter-examples the situation is exactly as described in our counter-example.

## 2 Choice of a suitable Hecke correspondence.

In this section we state a theorem that provides suitable candidates for the elements  $m \in G(\mathbb{Q}_l)$  to be used in the geometric criterion. The two important properties the element  $m$  has to have are

1. For any  $k_1$  and  $k_2$  in  $K_l$ ,  $k_1 m k_2$  generates an unbounded subgroup.
2. The degree of  $T_m$  is bounded by a uniform power of  $l$ .

The notations are as in the previous section. We have the following:

**Theorem 2.1** *Let  $l$  be a prime number splitting  $T$  and  $m$  an element of  $T(\mathbb{Q}_l)$ . We assume that the compact open subgroup  $K$  is of the form  $K = K^l \cdot K_l$ , where  $K^l$  is a compact open subgroup of  $G(\mathbb{A}_f^l)$  and  $K_l$  is a compact open subgroup of  $G(\mathbb{Q}_l)$  contained in an Iwahori subgroup  $I_l$  of  $G(\mathbb{Q}_l)$  in good position with respect to  $T$ . Let  $E$  be the reflex field of the Shimura datum defining  $V$ .*

*Then there exists an element  $m \in T(\mathbb{Q}_l)$  satisfying the following conditions :*

- (1)  $\text{Gal}(\overline{E}/E) \cdot V \subset V \cap T_m V$ .
- (2) *For every  $k_1, k_2 \in K_l$  the image of  $k_1 m k_2$  in  $G(\mathbb{Q}_l)$  generates an unbounded subgroup of  $G(\mathbb{Q}_l)$ .*
- (3)  $[K_l : K_l \cap m K_l m^{-1}] < l^k$  where  $k$  is a **uniform** integer.

The proof of the theorem is very technical and is omitted. Interested reader may consult section 8 of Klingler-Yafaev ‘The André-Oort conjecture’.

Let us say a few words about the first condition: the fact that  $m$  ‘comes from the Galois action’.

This, as next three lemmas show, can be achieved by replacing  $m$  satisfying conditions 2 and 3 by a uniform power. This does not destroy the conditions of the theorem.

As usual let  $C = H/H^{der}$  and  $E$  is the number field as before.

**Lemma 2.2** *There is a uniform integer  $n_1$  such that for any  $m \in T(\mathbb{Q}_l)$ , the power  $m^{n_1}$  is in the preimage of  $r_{(C, \{x\})}((\mathbb{Q}_l \otimes E)^*)$  in  $T(\mathbb{Q}_l)$  by the natural map  $T(\mathbb{Q}_l) \rightarrow C(\mathbb{Q}_l)$ .*

and :

**Lemma 2.3** *There is a uniform integer  $h$  such that the image of  $m^h$  in  $\overline{\pi_0}\pi(H)$  is in  $r_{(H, X_H)}(\text{Gal}(\overline{E}/E))$ .*

Let now  $Y$  be a subvariety of  $\text{Sh}_K(G, X)$  defined over  $E$  and containing  $V$  (special subvariety defined by  $\text{Sh}_{K_H}(H, X_H)$ ).

**Lemma 2.4** *With  $m$  as above :*

$$\text{Gal}(\overline{E}/E) \cdot V \subset V \cap T_{m^h}V$$

**Proof.** It suffices to see that some Galois conjugate of  $V$  is in  $T_{m^h}V$ .

The variety  $V$  is the image of  $(X_H^+, 1)$  in  $\text{Sh}_K(G, X)$ . Let  $\sigma$  be the element of  $\text{Gal}(\overline{F}/F)$  as above. By definition of the Galois action on the set of connected components of a Shimura variety, we get

$$\sigma(V) = \overline{(X_H^+, m^h)} \subset T_{m^h}V$$

where  $\overline{(X_H^+, m^h)}$  stands for the image of  $(X_H^+, m^h)$  in  $\text{Sh}_K(G, X)$ . □

### 3 End of the proof: the induction.

We use notations from lecture 2. We consider a Shimura datum  $(G, X)$  and  $K$  a compact open subgroup of  $G(\mathbb{A}_f)$  as in lecture two (in particular  $G$  is adjoint etc...) and  $S$  the image of  $X^+ \times \{1\}$  in  $\text{Sh}_K(G, X)$

For a special subvariety  $V$  defined by  $(H, X_H)$  with  $H = TH^{der}$ . Let, for simplicity of notations,  $\alpha_V = B^{i(T)}|K_T^m/K_T|$ ,  $L$  the splitting field of  $T$  and  $d_L$  the absolute value of its discriminant. We let  $F$  be a number field over which  $S$  admits a canonical model.

Recall the lower bound for the degree of the Galois orbit of  $V$ :

$$|\text{deg}(\text{Gal}(\overline{F}/F) \cdot V)| > C\alpha_V d_L^u$$

for some uniform constant  $C$ . The degree will always refer to the degree calculated with respect to the Baily-Borel line bundle.

**Theorem 3.1** *Let  $Y$  be a Hodge generic subvariety of  $\mathrm{Sh}_K(G, X)$  defined over  $F$ . We suppose that  $Y$  contains a non-strongly special subvariety  $V$  defined by Shimura datum  $(H, X_H)$ . We let  $T$  be the connected centre of  $H$ .*

*Suppose that there exists a prime  $l$  such that  $K_l$  is a maximal compact open subgroup of  $G(\mathbb{Q}_l)$ ,  $T_{\mathbb{F}_l}$  is a split torus and*

$$l^{(k+2f)2^r} \deg(Y)^{2^r} < C \alpha_V d_L^\mu$$

where  $r = \dim(Y) - \dim(V)$ .

*Then  $Y$  contains a special subvariety  $V'$  containing  $V$  properly.*

**Proof.** We only give a sketch here.

We denote  $d_Y := \deg_{\mathcal{L}_K}(Y)$ .

In order to apply our geometric characterisation, we need to pass to Iwahori level.

We let  $I_l$  be, as before, an Iwahori subgroup in good position with respect to  $T_{\mathbb{Q}_l}$  such that  $|K_l/I_l| < l^f$  and we let  $I$  be the compact open subgroup which is a product of  $K_p$ s for  $p \neq l$  and  $I_l$ .

The morphism  $\pi: \mathrm{Sh}_I(G, X) \rightarrow \mathrm{Sh}_K(G, X)$  is of degree  $\deg(\pi) \leq l^f$ .

As before,  $\mathcal{L}_I$  refers to the Baily-Borel line bundle on  $\mathrm{Sh}_I(G, X)$

Let  $\tilde{Y}$  be an irreducible component of  $\pi^{-1}Y$  defined over  $F$ . We have

$$\deg_{\mathcal{L}_I}(\tilde{Y}) \ll l^f \deg_{\mathcal{L}_K}(Y)$$

For simplicity of notations, we write  $Y$  and  $V$  for  $\tilde{Y}$  and  $\tilde{V}$ .

We let  $m$  be an element of  $T(\mathbb{Q}_l)$  given by the theorem and lemmas from the previous section. Recall in particular that the degree of  $T_m$  is bounded by  $l^k$ .

One notices that the varieties  $Y$  and  $T_m Y$  have a common component if and only if  $Y \subset T_m(Y)$  (this is because if they have a common component, then all its Galois conjugates are in the intersection).

Suppose that the intersection is proper (if it is not then we're already done). As  $T_m Y$  is defined over  $F$ , we have

$$\mathrm{Gal}(\overline{F}/F) \cdot V \subset Y \cap T_m Y$$

We have:

$$\deg(Y \cap T_m Y) \leq d_Y^2 [I_l : m I_l m^{-1}] < l^{k+2f} d_Y^2$$



We have to consider two cases:

Case 1:  $r > 1$ . We replace  $Y$  by an  $F$ -component of  $Y \cap T_m Y$  containing  $F$ . We also replace  $\text{Sh}_K(G, X)$  by the smallest special subvariety containing  $Y$  and pass to the adjoint Shimura datum. This is a much more delicate point than it looks! One has to guarantee that the image of  $V$  in this smaller special subvariety is still not strongly special.

In view of the above inequality, the condition of the theorem still holds and we reiterate what preceded.

Case 2:  $r = 1$ .

The inequality of the proposition shows that  $\deg(\text{Gal}(\overline{F}/F) \cdot V)$  exceeds the degree of  $Y \cap T_m Y$ , hence the intersection can not be proper. The geometric criterion allows to conclude.  $\square$

We can now finish the proof of the André-Oort conjecture, assuming the GRH.

Now, we suppose that  $Y$  is a subvariety of  $S$  containing a Zariski dense sequence  $Z_n$  of special subvarieties defined by  $(H_n, X_n)$  and that  $Y$  is defined over  $F$ . We write  $\alpha_n$  for  $\alpha_{Z_n}$  and  $d_n$  for  $d_{L_n}$  where  $L_n$  is the splitting field of the centre of  $H_n$ .

By replacing  $\text{Sh}_K(G, Y)$  with the smallest special subvariety containing  $Y$ , we assume that  $Y$  is Hodge generic.

If  $\alpha_n d_n$  is bounded, then the sequence  $Z_n$  is equidistributed and  $Y$  is special. Hence we suppose that  $\alpha_n d_n$  is unbounded.

In view of the previous theorem, the André-Oort conjecture will follow at once from the following:

**Theorem 3.2** *Assume GRH.*

*For any  $n \gg 0$ , there exists a prime  $l$  satisfying the following:*

1.  $T_{n, \mathbb{F}_l}$  is a split torus.
2.  $l^{(k+2f)2^r} \deg(Z)^{2^r} < C \alpha_n d_n^\mu$

**Proof.** Recall that  $\alpha_n = B^{i(T_n)} |K_{T_n}^m / K_{T_n}|$ . One has the following inequality:

$$\alpha_n \geq (Bc)^{i(T_n)} i(T_n)!$$

where  $c$  is a uniform constant.

To prove the existence of a prime  $l$  satisfying (1) and (2), we need to show that the number of primes split in  $L$  and satisfying (2) is larger than  $i_n$ . This is where effective Chebotarev theorem comes in.

This is a consequence of the fact that for a prime  $p$  unramified in  $L_n$  and such that  $K_{T_n,p}^m \neq K_{T_n,p}$ , one has

$$|K_{T_n,p}^m/K_{T_n,p}| \geq cp$$

Suppose that the function  $i_n := i(T_n)$  is unbounded. Then, classical inequalities (Stirling formula...) imply:

$$\alpha_n > (B' i_n)^{i_n}$$

where  $B'$  is uniform.

The effective Chebotarev theorem (see Lecture 2) gives

$$\pi_L(x) \gg \sqrt{x}$$

when  $x$  is large enough.

By taking  $x = (\frac{C}{\deg(Z)^{2r}} \alpha_n d_L^\mu)^{1/(k+2f2^r)}$  and using the inequality for  $\alpha_n$  written above, one easily concludes that  $\pi(x) > i_n$  when  $x$  is large enough. Hence we will find a prime with desired properties.

Assume now that  $i_n$  is bounded. Then an essentially trivial calculation shows again that  $\pi(x) > i_n$  when  $x$  is large enough.

□